

# Fluctuating hydrodynamics, current fluctuations and hyperuniformity in boundary-driven open quantum chains

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We consider a class of either fermionic or bosonic open quantum chains driven by dissipative interactions at the boundaries and study the interplay of coherent transport and dissipative processes, such as bulk dephasing and diffusion. Starting from the microscopic formulation, we show that the dynamics on large scales can be described in terms of fluctuating hydrodynamics (FH). This is an important simplification as it allows to apply the methods of macroscopic fluctuation theory (MFT) to compute the large deviation (LD) statistics of time-integrated currents. In particular, fermionic open chains display a third-order dynamical phase transition in LD functions. We show that this transition is manifested in a singular change in the structure of trajectories: while typical trajectories are diffusive, rare trajectories associated to atypical currents are ballistic and hyperuniform in their spatial structure. We confirm these results by numerically simulating ensembles of rare trajectories, via the cloning method, including the computation of their structure factors.

**Introduction.** There is much interest nowadays in understanding the collective macroscopic behaviour of non-equilibrium quantum systems that emerges from their underlying microscopic dynamics. This includes problems of thermalisation [1–5] and of novel non-ergodic [6] and driven phases [7] in quantum many-body systems, issues which also have started to be addressed experimentally [8–13]. Given that in practice interaction with an environment, while sometimes controllable, is always present, an important question is to what extent the interplay between coherent dynamics and dissipation influences the collective properties of such quantum non-equilibrium systems [14–20].

The hallmark of a driven non-equilibrium system is the presence of currents. Recently there has been important progress in the description of current bearing quantum systems with two complementary approaches. One corresponds to the study of quantum quenches where two halves of a system are prepared initially in different macroscopic states, and where the successive non-equilibrium evolution displays stationary bulk currents associated to transport of conserved charges [21–25]. Another corresponds to studies of one-dimensional spin chains coupled to dissipative reservoirs at the boundaries and described by quantum master equations [26–37]. While the transport depends on the precise nature of the processes present in the system, the studies above find that in appropriate long wavelength and long time limits the dynamics can be described in terms of effective hydrodynamic. A central question is to what extent the non-equilibrium dynamics of systems where there is an interplay between quantum coherent transport and dissipation displays behaviour similar to that of classical driven systems [38].

Here we address the statistics of currents in driven dissipative quantum systems and the spatial structure that

is associated with rare current fluctuations. We consider quantum chains for either fermionic or bosonic particles driven dissipatively through their boundaries. We also allow for dephasing and/or dissipative hopping in the bulk. Previous studies of similar systems have shown that with bulk dephasing and/or dissipative hopping transport is diffusive, while in the absence of both it is ballistic [28–31]. We show that the large scale dynamics of these systems in general admits a hydrodynamic description in terms of macroscopic fluctuation theory (MFT) [39]. In particular, we show that quantum stochastic trajectories can be described at the macroscopic level in terms of fluctuating hydrodynamics (FH) [40]. This macroscopic dynamics corresponds to that of the classical symmetric simple exclusion (SSEP) for fermionic chains, and to the classical symmetric inclusion processes (SIP) for bosonic chains [41–44].

The effective classical MFT/FH description in turn allows us to obtain the large deviation (LD) [45] statistics of time-integrated currents [38, 46, 47]. The cumulant generating function, or LD function, plays the role of a dynamical free-energy, and its analytic structure reveals the phase behaviour of the dynamics [48–58]. We find that in fermionic open chains the LD function displays a third-order transition between phases with very different transport dynamics. Similar LD criticality was found for classical SSEPs with open boundaries [59, 60]. We show that this transition is manifested in a singular change in the structure of the steady state, from diffusive for dynamics with typical currents, to ballistic and *hyperuniform* [61] (i.e., local density is anticorrelated and large scale density fluctuations are strongly suppressed) for dynamics with atypical currents.

**Models and effective hydrodynamics.** We consider a class of one-dimensional bosonic or fermionic quantum systems of  $L$  sites, weakly coupled to an environment,

and connected at its first and last sites to density reservoirs. In the Markovian regime, the evolution of an observable  $X$  follows a Lindblad form [62, 63],

$$\partial_t X_t = i[H, X_t] + \mathcal{D}^*[X_t], \quad (1)$$

where  $H$  is a tight-binding Hamiltonian with nearest-neighbour coherent particle hopping rate  $J$ ,

$$H = J \sum_{k=1}^{L-1} \left( a_{k+1}^\dagger a_k + a_k^\dagger a_{k+1} \right), \quad (2)$$

and  $a_k^\dagger, a_k$  denote (bosonic or fermionic) creation and annihilation operators acting on site  $k$ . The superoperator  $\mathcal{D}(\cdot) := \sum_\mu J_\mu(\cdot)J_\mu^\dagger - \frac{1}{2}\{J_\mu^\dagger J_\mu, (\cdot)\}$  is a Lindblad dissipator due to the coupling to the environment;  $J_\mu$  are *jump operators*, and  $\{\cdot, \cdot\}$  stands for the anticommutator. The dissipator has three contributions,  $\mathcal{D} := \mathcal{D}_1 + \mathcal{D}_L + \mathcal{D}_{\text{bulk}}$ . The first two corresponds to boundary driving terms where particles are introduced at site 1 and  $L$  with rates  $\gamma_{1,L}^{\text{in}}$  and removed with rates  $\gamma_{1,L}^{\text{out}}$ , cf. [33, 34],

$$\begin{aligned} \mathcal{D}_{1,L}^*[X] := & \gamma_{1,L}^{\text{in}} \left( a_{1,L} X a_{1,L}^\dagger - \frac{1}{2} \{a_{1,L} a_{1,L}^\dagger, X\} \right) + \\ & + \gamma_{1,L}^{\text{out}} \left( a_{1,L}^\dagger X a_{1,L} - \frac{1}{2} \{a_{1,L}^\dagger a_{1,L}, X\} \right). \end{aligned} \quad (3)$$

The third contribution accounts for dissipation in the bulk, including site dissipation with rate  $\gamma$  and dissipative hopping with rate  $\varphi$ , cf. [30, 31],

$$\begin{aligned} \mathcal{D}_{\text{bulk}}^*[X] := & \gamma \sum_{k=1}^L \left( n_k X n_k - \frac{1}{2} \{n_k^2, X\} \right) + \\ & + \frac{\varphi}{2} \sum_{k=1}^{L-1} \left( [[L_k^\dagger, X], L_k] + [[L_k, X], L_k^\dagger] \right), \end{aligned} \quad (4)$$

with  $n_k = a_k^\dagger a_k$  being the  $k$ -th site particle number operator, and  $L_k = a_{k+1}^\dagger a_k$ .

To derive the effective macroscopic description we consider the average occupation in each site  $\langle n_m \rangle_t$ , which from Eqs. (1-4) obeys a *continuity equation*,

$$\partial_t \langle n_m \rangle_t = -\langle (j_m^{\text{co}} - j_{m-1}^{\text{co}}) + (j_m^{\text{dis}} - j_{m-1}^{\text{dis}}) \rangle_t, \quad (5)$$

where  $j_m^{\text{co}} := -iJ(a_{m+1}^\dagger a_m - a_m^\dagger a_{m+1})$ , and  $j_m^{\text{dis}} := \varphi(n_m - n_{m+1})$ , are the two contributions to the current [31]:  $j_m^{\text{co}}$  is the coherent, and thus quantum in origin, particle current between  $m$  and  $m+1$ , while  $j_m^{\text{dis}}$  is the analogous of the *stochastic* current in SSEPs. The next step is to rescale space and time by suitable powers of the chain length  $L$  to get meaningful equations in the thermodynamic limit. As shown in [64] the correct rescaling is the diffusive one, with  $x := m/L \in [0, 1]$ , and  $\tau := t/L^2$ , for macroscopic space and time variables. For large  $L$  and finite  $\gamma$ , finite  $\tau$  means  $t\gamma \gg 1$ . In this regime, due the

action of  $\mathcal{D}$  on coherences,  $\mathcal{D}^*[a_m^\dagger a_n] = -(\gamma + 2\varphi)a_m^\dagger a_n$ , and its interplay with evolution due to the Hamiltonian, the quantum contribution to the current becomes  $j_m^{\text{co}} \approx \frac{2J^2}{\gamma + 2\varphi}(n_m - n_{m+1})$  [64]. This, together with the space and time rescaling, reduces Eq. (5) to,

$$\partial_\tau \rho_\tau(x) = D \partial_x^2 \rho_\tau(x), \quad (6)$$

where the macroscopic density is given by the average occupation by  $\rho_\tau(x) := \langle n_{m=xL} \rangle_{t=\tau L^2}$ , and the effective diffusion rate reads,

$$D := \left( \varphi + \frac{2J^2}{\gamma + 2\varphi} \right). \quad (7)$$

Dissipation at the chain ends, Eq. (3), set the boundary conditions on the density  $\rho_\tau(0) = \varrho_0$  and  $\rho_\tau(1) = \varrho_1$  for all  $\tau$ ,

$$\varrho_0 := \frac{\gamma_1^{\text{in}}}{\gamma_1^{\text{out}} \pm \gamma_1^{\text{in}}}, \quad \varrho_1 := \frac{\gamma_L^{\text{in}}}{\gamma_L^{\text{out}} \pm \gamma_L^{\text{in}}}, \quad (8)$$

where plus (minus) is for fermionic (bosonic) systems, restricted to the case  $\gamma_{1,L}^{\text{out}} - \gamma_{1,L}^{\text{in}} > 0$ . Equations (6), (7) and (8) encode the hydrodynamics description of a both fermionic and bosonic chains subject to dissipative drive whose macroscopic dynamics is diffusive. This description is a generalisation of previously studied special cases of this problem: for example, in the fermionic case, in the absence of dissipative hopping Eq. (7) reduces to the known value for that of the open  $XX$  chain with dephasing [34], while for dissipative hopping and no dephasing to that of [31]. The macroscopic dynamics Eqs. (6), (7) and (8) unifies these previous results, together with driven bosonic chains, clarifying the interplay between dephasing, coherent and stochastic hopping.

**Stochastic quantum trajectories and fluctuating hydrodynamics.** The diffusive character of the driven quantum chains opens up the possibility of applying MFT [39] for calculating the LD statistics of currents in our systems of interest. This is a huge simplification as it reduces the non-trivial task of computing the current LD function to a variational problem. Equations (6-8) provide an effective description at the macroscopic level of the exact microscopic master equation, Eqs. (1-4). In order to derive a MFT description we need an equivalent macroscopic description of the corresponding fluctuating quantum trajectories [69]. In the “input-output” formalism [69] the dynamics of a system operator  $X_t$  in terms of both the system and the environment is described in terms of a quantum stochastic differential equation (QSDE),

$$\begin{aligned} dX = & i[H, X]dt + \mathcal{D}^*(X)dt \\ & + \sum_\mu ([J_\mu^\dagger, X]dB_\mu + [X, J_\mu^\dagger]dB_\mu^\dagger), \end{aligned} \quad (9)$$

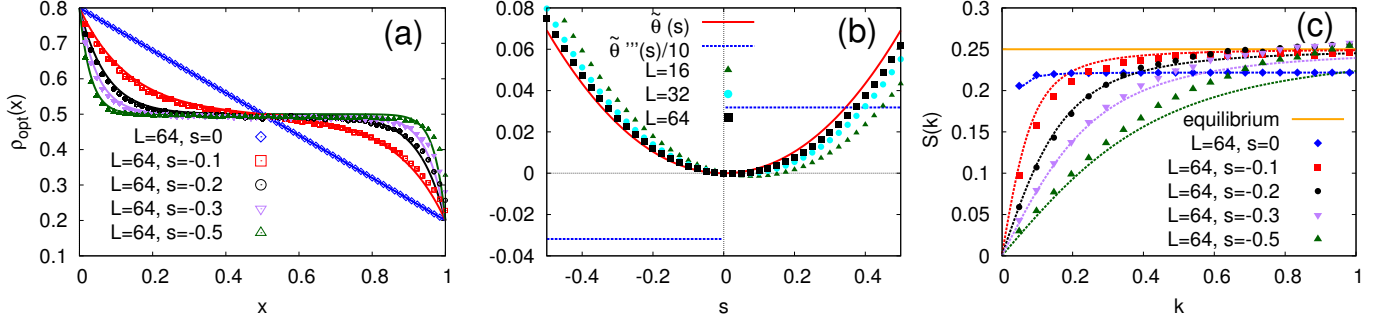


FIG. 1. (Color online) **Dynamical transition and hyperuniformity in the one-dimensional SSEP with open boundaries.** Here  $\varrho_0 = 0.8$ ,  $\varrho_1 = 0.2$  and  $D = 1$ . Numerical results (symbols) are obtained with the cloning method in continuous time with 1000 clones. This method generates rare trajectories efficiently by means of population dynamics techniques similar to those of quantum diffusion Monte Carlo [65–68]. (a) Analytical optimal profiles predicted by the MFT (solid lines) for four different values of  $\lambda = sL$  with  $L = 64$ . The numerical results (symbols) confirm that the profiles tend to  $\rho = 1/2$  as  $\lambda = sL$  increases. (b) LD function  $\tilde{\theta}(s)$  given by Eq. (14) (solid red line) together with its third derivative (dashed blue line). We can observe a good convergence of the numerical results to the theoretical prediction around  $|s| \ll 1$  for increasing system sizes. (c) Static structure factor  $S(k)$  for different values of the bias  $s$ , showing agreement between theory (dashed lines) and simulations (symbols), especially for small values of  $|s|$  which is the regime of validity of fluctuating hydrodynamics. Note that even for moderate values of  $s$  ( $s = -0.5$ ), a hyperuniform regime, with  $S(k) \rightarrow 0$  for small  $k$ , is still apparent from the simulations.

where  $\{J_\mu\}$  are the jump operators in the Lindblad master equation, cf. Eqs. (1-4), and  $dB_\mu, dB_\mu^\dagger$  are operators on the environment representing a quantum Wiener process and obeying the quantum Ito rules [69],  $(dB_\mu)^\dagger = (dB_\mu^\dagger)^\dagger = dB_\mu^\dagger$ ,  $dB_\mu^\dagger dB_\mu = 0$  and  $dB_\mu dB_\mu^\dagger = dt$ . Equation (9) makes explicit the quantum noise in the dynamics of the system, and tracing out over the environment (assumed to be in the vacuum for all modes  $\mu$ ) reduces Eq. (9) to the master equation Eq. (1).

By considering the QSDEs, Eq. (9), for the microscopic occupations  $n_m$  and currents  $j_m := j_m^{\text{co}} + j_m^{\text{dis}}$ , which forms a system of closed equations, and appropriately considering the change to macroscopic variables and fields (see Supplemental Material [64] for details), we obtain,

$$\partial_\tau \hat{\rho}_\tau(x) = D \partial_x^2 \hat{\rho}_\tau(x) - \partial_x \xi_\tau(x), \quad (10)$$

where  $\xi$  is a zero-mean Gaussian noise with covariance

$$\langle \xi_\tau(x) \xi_{\tau'}(x') \rangle = L^{-1} \sigma(\hat{\rho}_\tau(x)) \delta(x - x') \delta(\tau - \tau'), \quad (11)$$

and the *mobility* is given by [64],

$$\sigma(\rho) = 2D \rho(1 \mp \rho) \text{ fermions/bosons} \quad (12)$$

In Eq. (10)  $\hat{\rho}_\tau(x)$  and  $\hat{j}_\tau(x) := -D \partial_x \hat{\rho}_\tau(x) + \xi_\tau(x)$  indicate fluctuating density and current fields while  $\rho_\tau(x)$  and  $j_\tau(x)$  the noise-averaged ones. When averaging over the noise the Eq. (10) reduces to Eq. (6).

Equal densities at the boundaries corresponds to equilibrium conditions,  $\varrho_0 = \varrho_1 = \rho$ . At equilibrium we can compute the *compressibility* [64],  $\chi(\rho) = \rho(1 \mp \rho)$  (for fermions/bosons). Remarkably, this means that the *Einstein relation* [40] connecting the linear response of the density to a perturbation - the mobility - to its spontaneous fluctuations in equilibrium - the compressibility -

is obeyed,  $\sigma(\rho) = 2D\chi(\rho)$ . Here we have derived both  $\sigma(\rho)$  and  $\chi(\rho)$  analytically, but the validity of the Einstein relation was also perturbatively confirmed in the case of the driven XX chain in Ref. [34]. Note also that the forms of the mobility Eq. (12) are equivalent to the SSEP for fermions and to the SIP for bosons [41–44].

**Current fluctuations and large deviations phase transition.** We now compute the LD statistics of the empirical (i.e., time-averaged) total current,  $\hat{q} := T^{-1} \int_0^T dx \int_0^T d\tau \hat{j}_\tau(x)$ , up to a macroscopic time  $T = t/L^2$ . For long times  $t$  we expect its probability to have a LD form,  $P_t(q) := \langle \delta(q - \hat{q}) \rangle \approx e^{-t\phi(q)}$ , where  $\phi(q)$  is the LD *rate function*. The same information is encoded in the moment generating function (MGF),  $Z_t(s) := \int dq e^{-stq} P_t(q)$ , where  $s$  is the *counting field* conjugate to  $q$ . The MGF also has a LD form,  $Z_t(s) \approx e^{t\theta(s)}$ . The function  $\theta(s)$  is the *scaled cumulant generating function* (SCGF, sometimes also called simply LD function), and is related to  $\phi(q)$  via a Legendre transform [45]. The MGF plays the role of a dynamical partition sum, and can be written as a path-integral [70, 71],

$$Z_t = \int d\bar{\rho} d\rho e^{-L \int_0^1 dx \int_0^T d\tau \mathcal{L}_\lambda(\rho, \bar{\rho})}, \quad (13)$$

where  $\bar{\rho}$  is a response field, and the Lagrangian reads

$$\mathcal{L}_\lambda(\rho, \bar{\rho}) = i\bar{\rho} (\partial_\tau \rho - D \partial_x^2 \rho) - \lambda D \partial_x \rho - \frac{\sigma(\rho)}{2} (i\partial_x \bar{\rho} - \lambda)^2.$$

In the expressions above we have introduced a macroscopic counting field  $\lambda = Ls$  [59, 72], which can be thought of as the conjugate field to the spatially averaged empirical current per site. There are two regimes to consider. One is when  $\lambda$  is finite, which implies small  $s = O(L^{-1})$ ; this corresponds to the fluctuations of the spatially averaged empirical current. In this limit, the SCGF

can be obtained from Eq. (13) via a saddle-point approximation, giving  $\theta_{\text{MFT}}(\lambda) = (TL)^{-1} \log Z_{t=TL}^{\text{saddle}}(s = \lambda/L)$ , see [59] for details.  $\theta_{\text{MFT}}(\lambda)$  is analytic in  $\lambda$ , and there is no phase transition at the MFT level of description.

The second regime is that of small but finite  $s$ , and therefore large  $\lambda = O(L)$ , which probes dynamics associated to fluctuations of the empirical total current. To obtain the SCGF in this case we have to consider also the one loop correction to the saddle-point approximation when computing Eq. (13) before taking the long time/large size limits, cf. Ref. [59]. In terms of the original microscopic variables, the corresponding SCGF function for fermions (SSEP) is

$$\begin{aligned} \tilde{\theta}(s) &:= \lim_{L \rightarrow \infty} (tL)^{-1} \log Z_t^{\text{saddle+1loop}}(s) \\ &= \frac{\sigma s^2}{2} + \frac{D\sqrt{2}}{24\pi} \left| \frac{\sigma''\sigma}{D^2} \right|^{3/2} |s|^3 \end{aligned} \quad (14)$$

where  $\sigma = \sigma(\rho_{\text{opt}})$ , with  $\rho_{\text{opt}}$  the saddle-point solution of Eq. (13) and  $\sigma'' = -4D$ . Equation (14) is obtained assuming that the optimal profiles are time-independent (as proved in Ref. [73]). This we verify numerically by generating ensembles of rare trajectories via cloning [65–68], see Fig. 1(a). For bosons (SIP) the one loop correction is analytic in  $s$ , thus no phase transition is displayed.

Equation (14) has a discontinuity in its third-order derivative, and is also confirmed numerically for large  $L$ , see Fig. 1(b). Since the MFT description corresponds to the effective large scale dynamics of the driven quantum chains, this means that in the fermionic case there is a third-order dynamical phase transition occurring at  $s = 0$ .

**Ballistic dynamics and hyperuniformity for rare currents in fermionic chains.** As we now show, this transition at  $s = 0$  in the LD function, Eq. (14), corresponds to a change in the dynamics: while dynamics that leads to typical empirical currents is diffusive, that leading to atypical currents is ballistic and with hyperuniform spatial structure. This can be proved using MFT to calculate the dynamical structure factor for rare dynamics, cf. [74, 75].

The dynamical structure factor,  $S(k, t) := L^{-1} \langle \delta \tilde{n}_k(0) \delta \tilde{n}_k^*(t) \rangle$ , is defined in terms of the spatial Fourier transform,  $\delta \tilde{n}_k(t)$ , of the microscopic particle fluctuations,  $\delta n_m(t) := n_m(t) - \langle n_m \rangle_t$ . To measure  $S(k, t)$  for dynamics associated with  $s \neq 0$  we have to average using the  $s$ -ensemble [49], which in terms of the path integral representation, Eq. (13), means  $\langle \cdot \rangle_s = e^{-t\tilde{\theta}(s)} \int D\rho D\bar{\rho}(\cdot) e^{-L \int dx d\tau \mathcal{L}_{sL}[\rho, \bar{\rho}]}$  [64].

In the large  $L$  limit we can write  $S(k, t)$  in terms of macroscopic quantities,  $S(k, t) = L \mathcal{S}(p, \tau)$  with  $p = Lk$ , where  $\mathcal{S}(p, \tau) = \langle \delta \tilde{\rho}_0(p) \delta \tilde{\rho}_\tau^*(p) \rangle_s$  and  $\delta \tilde{\rho}_\tau(p)$  is the spatial Fourier transform of the fluctuations of  $\delta \rho_\tau(x)$  around the optimal profile  $\rho_{\text{opt}}(x)$  for a given value of  $s$ . To evaluate  $\mathcal{S}(p, \tau)$  we need the quadratic expansion of the Lagrangian,  $\mathcal{L}_2$ , in terms of  $\delta \rho$  and  $\delta \bar{\rho}$ . Since in general

the coefficients in  $\mathcal{L}_2$  are space dependent through the space dependence of  $\rho_{\text{opt}}(x)$ , the Gaussian integral is non-trivial [64]. However, in the equilibrium case,  $\varrho_0 = \varrho_1 = 1/2$ , we have that  $\rho_{\text{opt}}(x) = 1/2$ , the coefficients in  $\mathcal{L}_2$  become constant, and the integration needed for  $\mathcal{S}(p, \tau)$  becomes straightforward in terms of Fourier modes [64]. Remarkably, the same situation is encountered for the driven case,  $\varrho_0 \neq \varrho_1$ , for large  $|\lambda|$  (i.e. finite  $s$ ): regardless of the density at the boundaries, the optimal profile in the bulk tends to  $\rho(x) = 1/2$  (such that  $\sigma(\rho)$  is maximized) [59]. Simulations confirm this, see Fig. 1(a).

The dynamical structure factor can then be computed, to obtain (see [64] for details),

$$\mathcal{S}(p, \tau) = \frac{\sigma p^2}{L} \frac{\exp\left(-\frac{\tau}{2} \sqrt{4D^2 p^4 - 2\lambda^2 \sigma'' \sigma p^2}\right)}{\sqrt{4D^2 p^4 - 2\lambda^2 \sigma'' \sigma p^2}}$$

It follows that for typical dynamics,  $s = 0$ , at equilibrium with  $\varrho_0 = \varrho_1 = \frac{1}{2}$ , the structure factor is diffusive,  $S(k, t) \propto \exp(-Dk^2 t)$ . In contrast, and more interestingly, for  $s \neq 0$  and for any value of the density at the boundaries, we get,

$$S(k, t) \sim \frac{\sigma |k|}{\sqrt{-2s^2 \sigma'' \sigma}} \exp\left(-\frac{|k|t}{2} \sqrt{-2s^2 \sigma'' \sigma}\right), \quad (15)$$

for small  $k$ . This means two things: (i) Dynamics associated with empirical currents away from the typical value have ballistic dynamical scaling,  $t \approx L^z$  with dynamical exponent  $z = 1$ . (ii) For small  $k$  the structure factor vanishes as linearly in  $|k|$ , large-scale density fluctuations are suppressed and the system becomes spatially *hyperuniform* [61]. These analytic results are confirmed by numerical simulations of rare trajectories via cloning, see Fig. 1(c). This shows that for driven fermions the most efficient way to generate dynamics with atypical values of the current is by means of a singular change to a hyperuniform spatial structure, similarly to what occurs in the SSEP with periodic boundaries [74].

**Conclusions.** We established a direct connection between the microscopic dynamics of boundary driven quantum open chains an effective fluctuating hydrodynamics description at the macroscopic level. This effective description is equivalent to that of classical SSEPs or SIPs for the case of fermions or bosons, respectively. Using the tools of macroscopic fluctuation theory we have shown the existence of a dynamical phase transition from a diffusive phase of typical dynamics, to a ballistic and hyperuniform phase with atypical currents. It would be interesting to experimentally probe the predicted phase transition by monitoring current fluctuations in boundary-driven cold atomic lattice systems, e.g. via a variant of the experiment reported in Ref. [76].

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# Fluctuating hydrodynamics, current fluctuations and hyperuniformity in boundary-driven open quantum chains: Supplemental Material

## Effective macroscopic description of the quantum chain evolution

We present the derivation of the effective diffusive equation, Eq. (6) in the main text, governing the dynamics of the coarse-grained particle density profile of the quantum chain. First of all, we provide a relation that will be extensively used throughout the derivation: it can be checked that

$$\lim_{L \rightarrow \infty} L^2 \int_0^t du e^{-L^2 \gamma(t-u)} f_L(u) = \frac{1}{\gamma} f_\infty(t), \quad (\text{S1})$$

whenever  $\{f_L(u)\}_L$  is a sequence of bounded functions  $\forall u > 0$ , converging in  $L$ ; namely,  $L^2$  times the exponential converges weakly (under integration) to a Dirac delta  $\delta(t-u)$ .

Given the quantum master equation  $\partial_t X_t = i[H, X_t] + \mathcal{D}^*[X_t]$ , we are interested in deriving an effective dynamics for the average density of particles in the rescaled coordinates,  $\tau = t/L^2$ ,  $\frac{m}{L} \rightarrow x \in [0, 1]$ . The time-scale  $t/L^2$  can be simply obtained by multiplying the action of the generator by a factor  $L^2$ ,

$$\partial_\tau X_\tau = L^2 (i[H, X_\tau] + \mathcal{D}^*[X_\tau]).$$

Regarding the spatial dimension, coarse-graining consists in mapping the  $L$ -site chain onto a line  $\Lambda = [0, 1]$ , in such a way that the  $m$ -th site of the chain corresponds to the point  $\frac{m}{L}$  in  $\Lambda$ . In order to account for this geometric mapping, one has to notice that for large  $L$ , the spacing between sites in  $\Lambda$ , equal to  $\frac{1}{L}$ , becomes infinitesimal. We now introduce a continuous description; namely, we interpret  $a_m^\dagger$  as the creator of a particle in the one-dimensional box centred in  $\frac{m}{L}$  of width  $\frac{1}{L}$ . Mathematically, by means of the continuous fields  $\{\alpha_x, \alpha_x^\dagger\}_{x \in \Lambda}$ , one has

$$a_m^\dagger = \sqrt{L} \int_{U_m} dx \alpha_x^\dagger$$

where  $U_m$  is the domain of the box across the point  $\frac{m}{L} \in \Lambda$ ; the multiplying factor  $\sqrt{L}$  is needed to guarantee the commutation (anti-commutation) relations of the discrete bosonic (fermionic) operators, starting from the continuous ones for  $\alpha_x, \alpha_x^\dagger$ . Moreover, the integral can be approximated for large  $L$  by

$$a_m^\dagger \sim \frac{1}{\sqrt{L}} \alpha_{\frac{m}{L}}^\dagger. \quad (\text{S2})$$

With this at hand, we have all the ingredients to perform the hydrodynamic limit. We start considering the action of the nearest-neighbour Hamiltonian  $H = J \sum_{k=1}^{L-1} (a_{k+1}^\dagger a_k + a_k^\dagger a_{k+1})$  on quadratic operators

$$i [H, a_m^\dagger a_n] = iJ [\Delta_L^2 (a_m^\dagger) a_n - a_m^\dagger \Delta_L^2 (a_n)] , \quad (\text{S3})$$

with

$$\Delta_L^2(X_m) = X_{m+1} - 2X_m + X_{m-1}.$$

For a generic bulk site  $m$ , the rescaled time-derivative of the expectation of the number operator  $n_m = a_m^\dagger a_m$  can be cast in the following form

$$\partial_\tau \langle n_m \rangle_\tau = -L^2 (\langle j_m^{co} - j_{m-1}^{co} \rangle_\tau + \langle j_m^{dis} - j_{m-1}^{dis} \rangle_\tau) ; \quad (\text{S4})$$

the operator  $j_m^{co}$  has the meaning of a coherent current through the sites  $m, m+1$  and is defined as

$$j_m^{co} := -iJ (a_{m+1}^\dagger a_m - a_m^\dagger a_{m+1}) , \quad (\text{S5})$$

while

$$j_m^{dis} := \varphi (n_m - n_{m+1}) ,$$

is a current analogous to the one of classical stochastic models. To close the equation for the number operators, one needs to work on the current  $j_m^{co}$ . In particular, given the dissipator  $\mathcal{D}^*$  and by using (S3), its  $\tau$  time-derivative reads

$$\partial_\tau j_m^{co} = L^2 [2J^2 (n_m - n_{m+1}) + J^2 Q_m - \tilde{\gamma} j_m^{co}] , \quad (\text{S6})$$

with

$$Q_m = a_{m+2}^\dagger a_m + a_m^\dagger a_{m+2} - a_{m+1}^\dagger a_{m-1} - a_{m-1}^\dagger a_{m+1} , \quad (\text{S7})$$

and  $\tilde{\gamma} = \gamma + 2\varphi > 0$ . The term  $Q_m$  will be shown not to contribute in the hydrodynamic limit, but still its derivative needs to be considered:

$$\partial_\tau Q_m = L^2 (-\tilde{\gamma} Q_m + i[H, Q_m]) ; \quad (\text{S8})$$

we will study later the action of the Hamiltonian on this operator. Now, by formally integrating (S6) and (S8) and by substituting the result for  $Q_m$  into the time-evolution of the coherent current, we get

$$\langle j_m^{co} \rangle_\tau \sim -2J^2 L^2 \int_0^\tau du e^{-L^2 \tilde{\gamma}(\tau-u)} \Delta_L \langle n_m \rangle_u + L^4 \int_0^\tau du \int_0^u dv e^{-L^2 \tilde{\gamma}(\tau-v)} i \langle [H, Q_m] \rangle_v , \quad (\text{S9})$$

with  $\Delta_L n_m = n_{m+1} - n_m$  and where we have neglected exponentially decaying terms in  $L$ . Notice that, neglecting for the moment the term involving  $Q_m$ , taking the large  $L$  limit and recalling (S1), one obtains

$$\langle j_m^{co} \rangle_\tau = \frac{2J^2}{\gamma + 2\varphi} \langle n_m - n_{m+1} \rangle_\tau , \quad (\text{S10})$$

showing how due to dephasing the quantum current contributes to the total transport as an additional stochastic term. However, before obtaining (S10), one has to substitute the results of (S9) for  $j_m^{co}, j_{m-1}^{co}$ , in equation (S4). By doing this and rearranging terms we find

$$\partial_\tau \langle n_m \rangle_\tau = 2J^2 L^4 \int_0^\tau du e^{-L^2 \tilde{\gamma}(\tau-u)} \Delta_L^2 \langle n_m \rangle_u + \varphi L^2 \Delta_L^2 \langle n_m \rangle_\tau - iJ^2 L^6 \int_0^\tau \int_0^u dudv e^{-L^2 \tilde{\gamma}(\tau-v)} \langle P_m \rangle_v , \quad (\text{S11})$$

with  $P_m = [H, Q_m - Q_{m-1}]$ . Considering relation (S2) and noticing that  $P_m$  is quadratic in bosonic/fermionic operators, we can write the following differential equation for the expectation of  $\eta_x = \alpha_x^\dagger \alpha_x$

$$\partial_\tau \langle \eta_{\frac{m}{L}} \rangle_\tau = 2J^2 L^4 \int_0^\tau du e^{-L^2 \tilde{\gamma}(\tau-u)} \Delta_L^2 \langle \eta_{\frac{m}{L}} \rangle_u + \varphi L^2 \Delta_L^2 \langle \eta_{\frac{m}{L}} \rangle_\tau - iJ^2 L^6 \int_0^\tau \int_0^u dudv e^{-L^2 \tilde{\gamma}(\tau-v)} \langle \tilde{P}_{\frac{m}{L}} \rangle_v , \quad (\text{S12})$$

where  $\tilde{P}_{\frac{m}{L}}$  is the quadratic operator resulting from  $P_m$ , by replacing the discrete fields by the continuous ones. The term  $L^2 \Delta_L^2 \langle \eta_{\frac{m}{L}} \rangle_\tau$  represents a finite difference second derivative of the density, which in the large  $L$  limit with  $\frac{m}{L} \rightarrow x$  becomes

$$\lim_{L \rightarrow \infty} \varphi L^2 \Delta_L^2 \langle \eta_{\frac{m}{L}} \rangle_\tau = \varphi \partial_x^2 \langle \eta_x \rangle_\tau . \quad (\text{S13})$$

Similarly, taking also into account (S1), one has

$$\lim_{L \rightarrow \infty} 2J^2 L^4 \int_0^\tau du e^{-L^2 \tilde{\gamma}(\tau-u)} \Delta_L^2 \langle \eta_{\frac{m}{L}} \rangle_u = \frac{2J^2}{\tilde{\gamma}} \partial_x^2 \langle \eta_x \rangle_\tau .$$

It remains to show that the last term of the r.h.s of equation (S12) does not contribute in the large  $L$  limit. Firstly, one can check that the following quantity is bounded

$$\lim_{L \rightarrow \infty} J^2 L^4 \int_0^\tau du \int_0^u dv e^{-L^2 \tilde{\gamma}(\tau-v)} = C < \infty .$$

As a consequence the modulus of the last term in (S12)

$$I = \lim_{L \rightarrow \infty} \left| J^2 L^6 \int_0^\tau \int_0^u dudv e^{-L^2 \tilde{\gamma}(\tau-v)} \langle \tilde{P}_{\frac{m}{L}} \rangle_v \right| \quad (\text{S14})$$



can be bounded by

$$I \leq C \lim_{L \rightarrow \infty} \max_{\forall t > 0} \left\{ L^2 \left| \langle \tilde{P}_L^m \rangle_t \right| \right\}. \quad (\text{S15})$$

To understand the contribution of the operator  $\tilde{P}_L^m$ , one needs to go back to the operator  $Q_m$ . The latter is made of the product of operators spaced by two lattice sites; for example, by using equation (S3), one has

$$i[H, a_{m+2}^\dagger a_m] = iJ \left( \Delta_L^2(a_{m+2}^\dagger) a_m - a_{m+2}^\dagger \Delta_L^2(a_m) \right); \quad (\text{S16})$$

then, by multiplying by  $L^2$  and considering the spatial scaling one has

$$\lim_{L \rightarrow \infty} L^2 \langle \Delta_L^2(\alpha_{\frac{m+2}{L}}^\dagger) \alpha_{\frac{m}{L}} - \alpha_{\frac{m+2}{L}}^\dagger \Delta_L^2(\alpha_{\frac{m}{L}}) \rangle_t = (\langle \partial_x^2 \alpha_x^\dagger \alpha_x \rangle_t - \langle \alpha_x^\dagger \partial_x^2 \alpha_x \rangle_t). \quad (\text{S17})$$

Due to the spatial coarse-graining, all terms of  $[H, Q_m]$  give the same hydrodynamic contribution equal to (S17). Since  $Q_m$  is made of two terms with positive sign and another two with negative one, the net result for the hydrodynamic limit of  $\langle [H, Q_m] \rangle_\tau$  is zero. This implies that in the large  $L$  limit  $L^2 \langle \tilde{P}_L^m \rangle_t \rightarrow 0$ . Thus, by defining  $\rho_\tau(x) = \langle \eta_x \rangle_\tau$ , we have shown that (S12) reads

$$\partial_\tau \rho_\tau(x) = \left( \varphi + \frac{2J^2}{\gamma + 2\varphi} \right) \partial_x^2 \rho_\tau(x). \quad (\text{S18})$$

Such a differential equation needs to be provided with two boundary conditions; these are given by the extremal sites of the chain. For the expectation value of the number operator of the first site  $a_1^\dagger a_1$ , one has

$$\partial_\tau \langle a_1^\dagger a_1 \rangle_\tau = L^2 \left[ \gamma_1^{in} - (\gamma_1^{out} \pm \gamma_1^{in}) \langle a_1^\dagger a_1 \rangle_\tau + iJ \langle a_2^\dagger a_1 - a_1^\dagger a_2 \rangle_\tau + \varphi \langle a_2^\dagger a_2 - a_1^\dagger a_1 \rangle_\tau \right]. \quad (\text{S19})$$

where the plus is for fermionic systems while the minus for bosonic ones; in the latter case,  $\gamma_1^{out} - \gamma_1^{in} > 0$  is needed for the convergence of the expectations. Going to the continuous description in  $\Lambda$  and formally integrating the above equation we get, for large  $L$  and by neglecting exponentially decaying terms,

$$\langle \alpha_{\frac{1}{L}}^\dagger \alpha_{\frac{1}{L}} \rangle_\tau \sim \frac{\gamma_1^{in}}{\gamma_1^{out} \pm \gamma_1^{in}} + L^2 \int_0^\tau du e^{-L^2(\gamma_1^{out} \pm \gamma_1^{in})(\tau-u)} \left( iJ \langle \alpha_{\frac{2}{L}}^\dagger \alpha_{\frac{1}{L}} - \alpha_{\frac{1}{L}}^\dagger \alpha_{\frac{2}{L}} \rangle_u + \varphi \langle \alpha_{\frac{2}{L}}^\dagger \alpha_{\frac{2}{L}} - \alpha_{\frac{1}{L}}^\dagger \alpha_{\frac{1}{L}} \rangle_u \right). \quad (\text{S20})$$

By using (S1), and given that

$$\lim_{L \rightarrow \infty} \langle \alpha_{\frac{2}{L}}^\dagger \alpha_{\frac{1}{L}} - \alpha_{\frac{1}{L}}^\dagger \alpha_{\frac{2}{L}} \rangle_u = 0, \quad \lim_{L \rightarrow \infty} \langle \alpha_{\frac{2}{L}}^\dagger \alpha_{\frac{2}{L}} - \alpha_{\frac{1}{L}}^\dagger \alpha_{\frac{1}{L}} \rangle_u = 0,$$

one finds that the left boundary density is given by

$$\varrho_0 = \langle \alpha_0^\dagger \alpha_0 \rangle_\tau = \lim_{L \rightarrow \infty} \langle \alpha_{\frac{1}{L}}^\dagger \alpha_{\frac{1}{L}} \rangle_\tau = \frac{\gamma_1^{in}}{\gamma_1^{out} \pm \gamma_1^{in}}. \quad (\text{S21})$$

Similarly, at the right boundary one has

$$\varrho_1 = \langle \alpha_1^\dagger \alpha_1 \rangle_\tau = \frac{\gamma_L^{in}}{\gamma_L^{out} \pm \gamma_L^{in}}, \quad (\text{S22})$$

provided  $\gamma_L^{out} \pm \gamma_L^{in} > 0$ .

### Derivation of the fluctuating hydrodynamics

In order to apply the tools of fluctuating hydrodynamics to the quantum case, one needs to understand the dynamical description of the single realizations of the averaged dynamics implemented by Eq. (1) in the main text. Indeed, the Lindblad evolution (1) is obtained by tracing over the environment degrees of freedom, thus losing track of all possible realizations of the dynamics of the system plus environment. A way to unravel these single trajectories is offered by the “input-output” formalism, which makes explicit the presence of extra degrees of freedom representing the environment.

In the following, we will show how from the quantum stochastic master equation, one obtains the Langevin equation  $\partial_\tau \hat{\rho}_\tau(x) = -\partial_x \hat{j}_\tau(x)$ , for the fluctuating density  $\hat{\rho}_\tau(x)$ , with fluctuating current  $\hat{j}_\tau(x) = -D\partial_x \hat{\rho}_\tau(x) + \xi_\tau(x)$ , being  $\xi_\tau(x)$  a Gaussian white noise. For the sake of simplicity, we shall focus on the case with  $\gamma, J \neq 0$  and  $\varphi = 0$ , but the same result holds for  $\varphi \neq 0$ . Within the “input-output” framework [69], our starting point is provided by the following quantum stochastic master equation:

$$dX = (i[H, X] + \mathcal{D}^*[X])dt + \sqrt{\gamma} \sum_{k=1}^L ([n_k, X]dB_m(t) + dB_m^\dagger(t)[X, n_m]) ; \quad (\text{S23})$$

where  $dB_m(t), dB_m^\dagger(t)$  are environment operators, representing quantum Wiener processes and obeying the following quantum Ito rules

$$dB_m(t)dB_n^\dagger(t) = \delta_{m,n} dt, \quad dB_m^\dagger(t)dB_n^\dagger(t) = dB_m(t)dB_n(t) = dB_m^\dagger(t)dB_n(t) = 0.$$

Since we are interested in the evolution of the density we consider (S23) with  $X = n_m$ , yielding

$$dn_m = -(j_m^{co} - j_{m-1}^{co})dt. \quad (\text{S24})$$

Now we set out to derive the effective density evolution in the coarse-grained coordinates  $x = \frac{m}{L}$ ,  $\tau = t/L^2$ . Notice that the time-rescaling acts differently on the two increments  $dt, dB_m(t)$ , and one has

$$dt = L^2 d\tau, \quad dB_m(t) = L dB_m(\tau),$$

as in the usual Wiener processes. The equation for the number operator thus becomes

$$dn_m = -L^2(j_m^{co} - j_{m-1}^{co})d\tau; \quad (\text{S25})$$

in the same time-scaling we consider now the differential of the quantum current: recalling (S6), one has

$$dj_m^{co} \sim L^2 [2J^2(n_m - n_{m+1}) - \gamma j_m^{co}] + L\sqrt{\gamma} \sum_{k=1}^L ([n_k, j_m^{co}]dB_m(t) + dB_m^\dagger(t)[j_m^{co}, n_m]) ,$$

where we have neglected the term  $Q_m$  of (S7) that, as in the deterministic case of the previous section, can be shown not to contribute to the effective Langevin equation. By manipulating the noise term of the above equation it can be rewritten as

$$dj_m^{co} \sim L^2 [2J^2(n_m - n_{m+1}) - \gamma j_m^{co}] d\tau + L\sqrt{\gamma} J X_m dN_m(\tau),$$

with

$$dN_m(\tau) = -i(dB_{m+1}(\tau) - dB_{m+1}^\dagger(\tau)) + i(dB_m(\tau) - dB_m^\dagger(\tau)), \quad X_m = a_{m+1}^\dagger a_m + a_m^\dagger a_{m+1}. \quad (\text{S26})$$

Integrating the above differential equation for  $j_m^{co}$ , and substituting it in the equation (S25) one finds

$$dn_m = \frac{2J^2}{\gamma} L^2 \Delta_L^2(n_m) d\tau - \frac{J}{\sqrt{\gamma}} (X_m dN_m(\tau) - X_{m-1} dN_{m-1}(\tau)),$$

where we have used (S1) and neglected the exponentially decaying term in  $L$ . Moving to the continuous coordinate given by (S2), again with  $\eta_x = \alpha_x^\dagger \alpha_x$ , one gets

$$\partial_\tau \eta_x = \frac{2J^2}{\gamma} L^2 \Delta_L^2(\eta_x) - L\Delta_L \left( \frac{J}{\sqrt{L\gamma}} \tilde{X}_x \frac{d\nu_x(\tau)}{d\tau} \right), \quad (\text{S27})$$

with  $\tilde{X}_x$  being the analogous operator of  $X_m$  but written in terms of the continuous fields  $\alpha_x, \alpha_x^\dagger$ ;  $d\nu_x$  is also the analogous operator of  $dN_m(\tau)$ , but written in term of the coarse-grained environment operator  $d\beta_x(\tau), d\beta_x^\dagger(\tau)$ . In particular, the analogous relation to (S2) holds

$$dB_m^\dagger(\tau) \sim \frac{1}{\sqrt{L}} d\beta_x^\dagger(\tau),$$

which is responsible, for the extra factor  $\frac{1}{\sqrt{L}}$  in the round brackets of equation (S27). We see in (S27) the same deterministic diffusion term of (S18) (for  $\varphi = 0$ ) minus the first derivative of a noise term, that we denote by  $\xi_\tau(x)$ ,

$$\xi_\tau(x) = \frac{J}{\sqrt{L}\gamma} \tilde{X}_x \frac{d\nu_x(\tau)}{d\tau}.$$

Thus the evolution equation for the fluctuating density  $\hat{\rho}_\tau(x)$  is given by

$$\partial_\tau \hat{\rho}_\tau(x) = D \partial_x^2 \hat{\rho}_\tau(x) - \partial_x \xi_\tau(x),$$

where the noise term  $\xi_\tau(x)$  has a covariance

$$\langle \xi_\tau(x) \xi_{\tau'}(y) \rangle = \frac{\sigma(\hat{\rho}_\tau(x))}{L} \delta(x-y) \delta(\tau-\tau').$$

with  $\sigma(\rho) = 2D\rho(1 \pm \rho)$ , where the plus stands for bosons and the minus for fermions. This covariance can be derived by directly computing  $\langle \xi_\tau(x) \xi_{\tau'}(y) \rangle$

$$\langle \xi_\tau(x) \xi_{\tau'}(y) \rangle = \left\langle \frac{J}{\sqrt{L}\gamma} \tilde{X}_x \frac{d\nu_x(\tau)}{d\tau} \frac{J}{\sqrt{L}\gamma} \tilde{X}_y \frac{d\nu_y(\tau')}{d\tau'} \right\rangle = \frac{J^2}{L\gamma} \tilde{X}_x \tilde{X}_y \left\langle \frac{d\nu_x(\tau)}{d\tau} \frac{d\nu_y(\tau')}{d\tau'} \right\rangle.$$

One can check that  $\left\langle \frac{d\nu_x(\tau)}{d\tau} \frac{d\nu_y(\tau')}{d\tau'} \right\rangle = 2\delta(\tau-\tau')\delta(x-y)$ . To understand the remaining part, we look at the square of  $X_m$

$$X_m^2 = a_{k+1}^\dagger a_k a_{k+1}^\dagger a_k + a_k^\dagger a_{k+1} a_k^\dagger a_{k+1} + n_{k+1}(1 \pm n_k) + n_k(1 \pm n_{k+1});$$

due to the presence of dephasing damping quantum coherences on fastest time-scales than those of  $\tau$ , we assume a local “thermal” equilibrium state for the infinitesimal domain across the bond  $x = \frac{m}{L}$ . This implies that the first two terms of the r.h.s in the above equation are zero. Hence, the coarsed-grained operator  $\tilde{X}_x^2$  reads

$$\tilde{X}_x^2 = 2\hat{\rho}_\tau(x) (1 \pm \hat{\rho}_\tau(x)),$$

where the plus stands for bosons and the minus for fermions, and with  $\hat{\rho}_\tau(x)$  being the fluctuating particle density. Thus, we have found, starting from the quantum stochastic master equation describing the quantum trajectories of the microscopic evolution, the equation governing the fluctuating hydrodynamics in the coarse-grained macroscopic description; this reads

$$\partial_\tau \hat{\rho}_\tau(x) = -\partial_x \hat{j}_\tau(x), \quad \hat{j}_\tau(x) = -D \partial_x \hat{\rho}_\tau(x) + \xi_\tau(x),$$

with  $D = \frac{2J^2}{\gamma}$ , and  $\xi_\tau(x)$  a noise with properties already discussed. The same result holds by considering  $\varphi \neq 0$ , i.e. with  $D = \varphi + \frac{2J^2}{\gamma+2\varphi}$ .

### Computation of $\chi(\rho)$

To compute the compressibility  $\chi(\rho)$ , defined as the density fluctuations in equilibrium, one needs to know the stationary state for  $\varrho_0 = \varrho_1$ ; this means to consider a dissipator  $\mathcal{D}^*$  with boundary term rates such that

$$\frac{\gamma_1^{in}}{\gamma_1^{out} \pm \gamma_1^{in}} = \frac{\gamma_L^{in}}{\gamma_L^{out} \pm \gamma_L^{in}},$$

which implies that  $\frac{\gamma_1^{in}}{\gamma_1^{out}} = \frac{\gamma_L^{in}}{\gamma_L^{out}}$ . In the Schrödinger picture, where states are evolved, the stationary state is defined as the quantum state  $R$ , such that

$$-i[H, R] + \mathcal{D}[R] = 0;$$

the strategy is to consider an ansatz form for the quantum equilibrium state and show that it satisfies the above relation. We thus consider

$$R \propto \prod_{n=1}^L e^{-\beta a_n^\dagger a_n};$$

notice that  $\beta$  here is a parameter that has to be adjusted in such a way that for every site  $h$ ,

$$\rho := \text{Tr}[R a_h^\dagger a_h] = \frac{e^{-\beta}}{1 \pm e^{-\beta}} = \frac{\gamma_1^{in}}{\gamma_1^{out} \pm \gamma_1^{in}},$$

where the plus is for fermions and the minus for bosons. Therefore, one chooses  $\beta$  to be  $\beta = \log\left(\frac{\gamma_1^{out}}{\gamma_1^{in}}\right)$ . Moreover, the state  $R$  can be written as an exponential of the total number of particles

$$R \propto e^{\beta N}, \quad N = \sum_{n=1}^L a_n^\dagger a_n.$$

Given that all jump operators in  $\mathcal{D}_{Bulk}^*$ ,  $\{n_k, a_k^\dagger a_{k+1}, a_{k+1}^\dagger a_k\}$ , commute with the total number of particles, one has  $\mathcal{D}_{Bulk}[R] = 0$ ; furthermore, also the Hamiltonian  $H$  is such that  $[H, R] = 0$ . Thus, we just have to check the action of the boundary terms  $\mathcal{D}_{1,L}$  on the ansatz state  $R$ .  $\mathcal{D}_1$  and  $\mathcal{D}_L$  act only on the first, respectively last site of the chain, so that their action on  $R$  is

$$\mathcal{D}_1[e^{\beta N}] = \mathcal{D}_1[e^{\beta a_1^\dagger a_1}] \prod_{n=2}^L e^{\beta a_n^\dagger a_n}, \quad \mathcal{D}_L[e^{\beta N}] = \prod_{n=1}^{L-1} e^{\beta a_n^\dagger a_n} \mathcal{D}_L[e^{\beta a_L^\dagger a_L}].$$

We focus on  $\mathcal{D}_1$ , being the computation also valid for the last site of the chain. Straightforward calculations give

$$\mathcal{D}_1[e^{\beta a_1^\dagger a_1}] = e^{-\beta a_1^\dagger a_1} \left[ (\gamma_1^{in} e^{\beta} - \gamma_1^{out}) a_1^\dagger a_1 + (\gamma_1^{out} e^{-\beta} - \gamma_1^{in}) a_1 a_1^\dagger \right],$$

which, given the choice of the parameter  $\beta$ , is identically zero. Thus, the state  $R$  is exactly the stationary state of the considered dynamics at equilibrium. With this, we can readily compute of the compressibility.

Recalling its definition [40], for the state  $R$  one has

$$\chi(\rho) := \sum_{k=1}^L (\text{Tr}[R n_k n_1] - \rho^2);$$

thus, one gets

$$\chi(\rho) = \rho(1 \pm \rho), \tag{S28}$$

where the plus is for bosons while the minus for fermions.

### Derivation of the structure factor

In this section we provide details on the computation of both the static and the dynamical structure factor presented in the main text. We start defining the fluctuations around the average occupation numbers in the  $s$ -ensemble  $\delta n_m(t) = n_m(t) - \langle n_m(t) \rangle_s$ . Subsequently, by taking into account the open geometry of the system at hand, we also introduce their discrete Fourier sine-transform  $\delta \tilde{n}_k(t)$  (with  $k = \frac{\pi}{L}r$ ,  $r = 1, 2, \dots, L-1$ ),

$$\delta \tilde{n}_k(t) = \sqrt{2} \sum_{h=1}^L \sin(k h) \delta n_h(t).$$

By substituting this in the definition of the static structure factor,  $S(k) = \frac{1}{L} \langle |\delta \tilde{n}_k(t)|^2 \rangle_s$ , one obtains

$$S(k) = \frac{2}{L} \sum_{\ell, h=1}^L \sin(k \ell) \sin(k h) C_{\ell h}, \tag{S29}$$

with  $C_{hk} = \langle \delta n_h(t) \delta n_k(t) \rangle_s$  being the second cumulant of densities at site  $h$  and  $k$  computed in the  $s$ -ensemble; explicitly

$$C_{\ell h} = \langle n_\ell(t) n_h(t) \rangle_s - \langle n_\ell(t) \rangle_s \langle n_h(t) \rangle_s.$$

Once these correlations are known, the computation of the structure factor (S29) is straightforward. While at this microscopic level, analytical control of all density-density correlation functions is often out of reach, we shall show that in the present case one can still derive a closed simple formula for the structure factor, by exploiting the macroscopic scale of fluctuating hydrodynamics. The connection between these two levels of description is offered by the equality  $\langle n_m \rangle_t = \rho_\tau(x)$ , with  $x = \frac{m}{L} \in [0, 1]$  and  $t = L^2\tau$  (equivalently for deviations  $\delta n_m(t) = \delta \rho_\tau(x)$ ). In the large  $L$  limit, one can approximate summations with integrals, obtaining the following relation

$$\delta \tilde{n}_k(t) = \sqrt{2}L \int_0^1 dx \sin(px) \delta \rho_\tau(x) = L \delta \tilde{\rho}_\tau(p),$$

with  $p = Lk$ , and  $\delta \tilde{\rho}_\tau(p)$  being the Fourier sine transform of the density field  $\delta \rho_\tau(x)$ . This yields an expression of the structure factor in terms of macroscopic coarse-grained quantities,

$$S(k) = L\mathcal{S}(p), \quad \mathcal{S}(p) = \langle |\delta \tilde{\rho}_\tau(p)|^2 \rangle_s.$$

The  $s$ -ensemble expectation of density dependent observables can be computed in a path integral formalism as

$$\langle O \rangle_s = \frac{\int D\rho D\bar{\rho} O[\rho] e^{-L \iint dx d\tau \mathcal{L}_{sL}[\rho, \bar{\rho}]}}{\int D\rho D\bar{\rho} e^{-L \iint dx d\tau \mathcal{L}_{sL}[\rho, \bar{\rho}]}}.$$

Moreover, in the fluctuating hydrodynamics regime, i.e.  $|s| \ll 1$ , and small fluctuations  $\delta \rho_\tau(x)$ , these averages can be evaluated just considering contributions coming from a truncated second order Lagrangian,

$$\langle O \rangle_s = \frac{\int D\rho D\bar{\rho} O[\delta \rho] e^{-L \iint dx d\tau \mathcal{L}_2[\delta \rho, \delta \bar{\rho}]}}{\int D\rho D\bar{\rho} e^{-L \iint dx d\tau \mathcal{L}_2[\delta \rho, \delta \bar{\rho}]}} \quad (S30)$$

where

$$\mathcal{L}_2[\delta \rho, \delta \bar{\rho}] = i\delta \bar{\rho} (\partial_\tau \rho - D\partial_x^2 \delta \rho) + \frac{\sigma}{2} (\partial_x \delta \bar{\rho})^2 - \frac{\sigma''}{4} (i\partial_x \bar{\rho}_{opt} - \lambda)^2 (\delta \rho)^2 - i\sigma' (i\partial_x \bar{\rho}_{opt} - \lambda) \delta \rho \partial_x \delta \bar{\rho}. \quad (S31)$$

Even if the Lagrangian  $\mathcal{L}_2$  is quadratic in the fields, the computation of (S30) is still a difficult task, because of the presence of space-dependent coefficients. Remarkably, in the large  $|\lambda|$  regime one has that the stationary optimal profile  $\rho_{opt}(x)$  tends to the value  $\rho_{opt} \rightarrow 1/2$  almost everywhere, except for vanishingly small regions at the boundaries. As a consequence  $\sigma(\rho_{opt}) \rightarrow \sigma = 1/2$ , so that  $\sigma' \rightarrow 0$ . Moreover  $\bar{\rho}_{opt}$  is such that  $\partial_x \bar{\rho}_{opt} \rightarrow 0$  [59]. Thus, one can approximate  $\mathcal{L}_2$  with

$$\mathcal{L}_2 \sim \hat{\mathcal{L}}_2 = i\delta \bar{\rho} (\partial_\tau \rho - D\partial_x^2 \delta \rho) + \frac{\sigma}{2} (\partial_x \delta \bar{\rho})^2 - \frac{\sigma''}{4} \lambda^2 (\delta \rho)^2, \quad (S32)$$

with  $\mathcal{L}_2 = \hat{\mathcal{L}}_2$  only in the  $L \rightarrow \infty$  limit, for  $s \neq 0$ . Notice that  $\hat{\mathcal{L}}_2$  is the exact second order expansion of the Lagrangian  $\mathcal{L}$  in the equilibrium case  $\varrho_0 = \varrho_1 = 1/2$ . Therefore, for large  $L$ , expectation (S30) is well approximated by

$$\langle O \rangle_s \sim \frac{\int D\rho D\bar{\rho} O[\delta \rho] e^{-L \iint dx d\tau \hat{\mathcal{L}}_2[\delta \rho, \delta \bar{\rho}]}}{\int D\delta \rho D\bar{\rho} e^{-L \iint dx d\tau \hat{\mathcal{L}}_2[\delta \rho, \delta \bar{\rho}]}} \quad (S33)$$

which can be computed. Indeed, through the space-time Fourier expansion,

$$\delta \rho_\tau(x) = \frac{\sqrt{2}}{T} \sum_\omega \sum_{p>1} \sin(px) e^{-i\omega\tau} \delta \tilde{\rho}_{p,\omega}$$

with  $\omega = \frac{2\pi}{T}u$ ,  $u \in \mathbb{Z}$ , and, the equivalent one for  $\delta \bar{\rho}_\tau(x)$ , one can diagonalise  $\hat{\mathcal{L}}_2$ , obtaining

$$\iint dx d\tau \hat{\mathcal{L}}_2 = \frac{1}{T} \sum_{p>1, \omega \geq 0} \delta \tilde{\rho}_{p,\omega}^\dagger \cdot K_{p,\omega} \cdot \delta \tilde{\rho}_{p,\omega},$$

where  $\delta \tilde{\rho}_{p,\omega}^\dagger = (\delta \tilde{\rho}_{p,\omega}, \delta \tilde{\rho}_{p,\omega}^{tr})^{tr}$  (with  $a^{tr}$  denoting vector transposition and  $a^\dagger = (a^*)^{tr}$ ), and

$$K_{p,\omega} = \begin{pmatrix} -\frac{\sigma''\lambda^2}{2} & \omega + iDp^2 \\ -\omega + iDp^2 & \sigma p^2 \end{pmatrix}.$$

At this point, the computation of (S30) is reduced to evaluations of Gaussian path integrals. In terms of the Fourier fields  $\delta\tilde{\rho}_{p,\omega}$ ,

$$\delta\tilde{\rho}_\tau(p) = \frac{1}{T} \sum_{\omega} e^{-i\omega\tau} \delta\tilde{\rho}_{p,\omega}, \quad (\text{S34})$$

one can write

$$\mathcal{S}(p) = \frac{1}{T^2} \sum_{\omega, \omega'} e^{-i\tau(\omega - \omega')} \langle \delta\tilde{\rho}_{p,\omega} \delta\tilde{\rho}_{p,\omega'}^* \rangle_s.$$

Then, evaluating the above  $s$ -ensemble expectation with (S33), one gets

$$\langle \delta\tilde{\rho}_{p,\omega} \delta\tilde{\rho}_{p,\omega'}^* \rangle_s \sim \delta_{\omega, \omega'} \frac{T}{L} \frac{\sigma p^2}{\omega^2 + D^2 p^4 - \frac{\lambda^2 \sigma'' \sigma p^2}{2}}.$$

By replacing the summation over  $\omega$  with an integration in the long-time limit,  $\mathcal{S}(p)$  reads

$$\mathcal{S}(p) \sim \frac{1}{L} \frac{\sigma p^2}{\sqrt{4D^2 p^4 - 2\lambda^2 \sigma'' \sigma p^2}}.$$

Recalling that  $\lambda = sL$ ,  $p = Lk$ , and that in the large  $L$  limit approximations (S32)-(S33) become exact, one has

$$S(k) = \frac{\sigma k^2}{\sqrt{4D^2 k^4 - 2s^2 \sigma'' \sigma k^2}}. \quad (\text{S35})$$

Following the same strategy, we can also compute the dynamical structure factor, defined as

$$S(k, t) = \frac{1}{L} \langle \delta\tilde{n}_k(0) \delta\tilde{n}_k^*(t) \rangle_s = L \mathcal{S}(p, \tau), \quad \mathcal{S}(p, \tau) = \langle \delta\tilde{\rho}_0(p) \delta\tilde{\rho}_\tau^*(p) \rangle_s.$$

By means of the time Fourier transform Eq. (S34), one has

$$\mathcal{S}(p, \tau) = \frac{1}{T^2} \sum_{\omega, \omega'} e^{i\omega\tau} \langle \delta\tilde{\rho}_{p,\omega'} \delta\tilde{\rho}_{p,\omega}^* \rangle_s.$$

By using the previous  $s$ -ensemble expectations, the dynamical structure factor reads

$$\mathcal{S}(p, \tau) = \frac{1}{LT} \sum_{\omega} e^{i\omega\tau} \frac{\sigma p^2}{\omega^2 + D^2 p^4 - \frac{\lambda^2 \sigma'' p^2}{2}},$$

which in the large  $T$  limit can be computed as the following integral

$$\mathcal{S}(p, \tau) = \frac{1}{2\pi L} \int_{-\infty}^{\infty} d\omega e^{i\omega\tau} \frac{\sigma p^2}{\omega^2 + D^2 p^4 - \frac{\lambda^2 \sigma'' p^2}{2}}.$$

Therefore, one finds

$$\mathcal{S}(p, \tau) = \frac{1}{L} \frac{\sigma p^2}{\sqrt{4D^2 p^4 - 2\lambda^2 \sigma'' \sigma p^2}} \exp\left(-\frac{\tau}{2} \sqrt{4D^2 p^4 - 2\lambda^2 \sigma'' \sigma p^2}\right).$$

It is worth noting that in the equilibrium case,  $\varrho_0 = \varrho_1 = 1/2$ , for  $s = 0$ , this gives

$$S(k, t) = \frac{\sigma}{2D} \exp(-Dk^2 t),$$

since one has  $\tau L^2 = t$ . The exponent  $Dk^2 t$  displays a dynamical exponent  $z$ ,  $k^z t$  equal to  $z = 2$ . However, when considering biased trajectories ( $s \neq 0$ ), regardless of the values at the boundaries, one gets

$$S(k, t) \sim \frac{\sigma |k|}{\sqrt{-2s^2 \sigma'' \sigma}} \exp\left(-\frac{|k|t}{2} \sqrt{-2s^2 \sigma'' \sigma}\right),$$

for  $k \ll 1$ , showing a dynamical exponent  $z = 1$ .